Multifractal scaling of moments of mean first-passage time in the presence of Sinai disorder

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(Received 19 April 1995)

We consider mean first-passage time (MFPT) of random walks from one end to the other of a segment of a Sinai lattice. In the limit of the length of the lattice segment going to infinity, the distribution of MFPT over Sinai disorder has unbounded moments. We present a multifractal characterization of the distribution. We derive an analytical expression for the fractal dimension as a function of the strength of the disorder. We demonstrate that the multifractality of the limiting distribution manifests itself as self-similar fluctuations of the MFPT from one disorder configuration to the other.

PACS number(s): 05.40.+j, 05.60.+w, 61.43.Hv

I. INTRODUCTION

Distributions with unbounded moments arise in a variety of contexts, such as voltage and current fluctuations in random resistor networks [1], growth by diffusion limited aggregation [2], growth in random multiplicative environments [3], anomalous diffusion in disordered [4–6] and hierarchical structures [7], random river networks [8], ion collision cascades [9], return probabilities of simple random walks [10], etc., to name only some of them. A general question we address in this paper is the relation of such distributions to multifractal measures. To focus on the issue, let us consider a positive definite random variable t, and denote its distribution by $\rho(t, T)$, where T is a scaling parameter, which is related to the system size N, cf. Eq. (3). Let us consider the distribution $\rho(t, T)$ in the limit $T \to \infty$. The positive integer moments of $\rho(t, T)$ all diverge in this limit. Here our aim is to characterize the distribution $\rho(t, T)$ employing multifractal formalisms.

Let $M(Q, T)$ denote the Qth moment, defined formally as $M(Q, T) = \int_0^\infty t^Q \rho(t, T) dt$. We are interested in distributions with the property

$$M(Q, T) \overset{T \to \infty}{\sim} T^{\xi(Q)}, \tag{1}$$

where $\xi(Q)$ are the exponents characterizing the divergence of the moments in the limit $T \to \infty$. All the phenomena [1–10] referred to in the beginning have this property. If $\xi(Q)$ is a linear function of Q, we have a simple scaling with a single gap exponent $\Delta = \xi(Q) - \xi(Q - 1)$, as occurs in critical phenomena. Various systems investigated in recent times [1–10] have lead to nonlinear dependence of $\xi(Q)$ on Q, and hence require an infinity of gap exponents to characterize the distributions comprehensively.

In this paper we derive an explicit relation between $\xi(Q)$ and the multifractal exponents, normally denoted by $\tau(Q)$. This is accomplished by investigating the distribution of mean first-passage time (MFPT) of random walks on a lattice with Sinai disorder [11]. Such a derivation has become possible because of several simplifying features of the model considered. First, an exact expression for the average MFPT is available in terms of the quenched jump probabilities and the size of the lattice segment. Second, the dichotomic model of the Sinai disorder permits an exact enumeration of all the possible disorder configurations. Third, the simplicity of the model suggests a natural choice of the scale parameter as the largest value of MFPT, T, for which we have an exact analytical expression. As a specific application we show how one can calculate the fractal dimension $D(0)$ by considering the scaling behavior of T, and of the number of the disorder configurations as the size of the lattice segment goes to infinity.

The paper is organized as follows. In Sec. II we describe the model system studied, which consists of random walks on a finite segment of a lattice with site dependent random jump probabilities that obey the Sinai condition. We calculate the moments of the distribution of the MFPT over the disorder and characterize their divergence through appropriate scaling exponents, $\xi(Q)$. In Sec. III we construct a normalized partition function and numerically establish a scaling ansatz that gives the multifractal exponents $\tau(Q)$. We employ the multifractal formalism [12,13] and characterize the MFPT distribution through the scaling exponents $\tau(Q)$, the generalized Renyi dimensions $D(Q)$, and the spectrum of singularities $f(\alpha)$. In Sec. IV we state the relation between the exponents $\xi(Q)$ and $\tau(Q)$. We consider the case with $Q = 0$ and obtain an analytical expression for the fractal dimension as a function of the strength of disorder. We compare our analytical results with the numerical results of an earlier study [14], and find good agreeement. In Sec. V we demonstrate that the multifractal character of the distributions with unbounded moments manifests itself

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as statistically self-similar fluctuations of the randomly sampled values of the random variable from its distribution. In Sec. VI, we summarize briefly the results and the principal conclusions.

II. RANDOM WALKS ON A SINAI LATTICE

We consider a segment of a lattice with the sites labeled by $i = 0, 1, \ldots N$. The random walk at site j can jump to the site $j + 1$ with probability p_j, or to the site $j - 1$ with probability $1 - p_j$, in one step. At site 0, we have the right jump probability p_0, and a sojourn probability $1 - p_0$. The site N is absorbing. $\{p_i; i = 0, N - 1\}$ constitute a set of identically distributed independent random variables, and the common distribution obeys the Sinai condition [11], namely, the random variable $\ln[(1 - p)/p]$ has zero mean and finite variance, the latter denoted by σ^2. We consider a dichotomic model of the Sinai disorder by prescribing p to be $\frac{1}{2} \pm \epsilon$ with equal probabilities. The parameter ϵ measures the strength of disorder, and can take values between 0 and 1/2. The larger the value of ϵ, the stronger is the disorder. $\epsilon = 0$ corresponds to simple random walks. It is easily verified that the above prescription for the distribution of p obeys the Sinai condition.

We consider a given realization of the random lattice, described by a fixed set $\{p_i; i = 0, N - 1\}$. The random walk starts at 0, and eventually gets absorbed at N. Let $t_{0,N}$ denote the number of steps a random walk takes to reach the right boundary for the first time. Let $t = \langle t_{0,N} \rangle$, where the angular brackets denote an average over all possible random walks, for a given realization of the quenched jump probabilities. An exact expression for t - the MFPT, in terms of the random jump probabilities can be derived, employing the generating function technique [15–17], and it is given by

$$t = \sum_{k=0}^{N-1} \frac{1}{p_k} + \sum_{k=0}^{N-2} \frac{1}{p_k} \sum_{i=k+1}^{N-1} \prod_{j=k+1}^{i} \frac{1-p_j}{p_j}. \qquad (2)$$

Let us denote by $\{t\}$ the average MFPT. The curly brackets denote the average taken over the disorder, i.e., over all possible realizations of the set of random variables $\{p_i; i = 0, N - 1\}$. In the asymptotic limit of $N \to \infty$, $\{t\}$ diverges as $\beta^N(\epsilon)$; see [15], where $\beta(\epsilon) \equiv \langle (1-p)/p \rangle = (1+4\epsilon^2)/(1-4\epsilon^2)$, for the dichotomic model of Sinai disorder. The typical value of MFPT, defined as $\exp[\langle \ln t_{0,N} \rangle]$, however, diverges slower, as $\exp[\sigma\sqrt{\pi/2}N^{1/2}]$; see [5]. This implies that the distribution of t over Sinai disorder has a power-law tail; indeed, earlier studies [6,17] have shown that asymptotically the distribution has a $1/t$ tail.

For the dichotomic model of the Sinai disorder we observe that given a value of N, there are 2^N possible realizations of the random lattice and the value of t for each of these can be calculated exactly employing the expression (2). It is easily seen that the largest value of t obtains when at all the sites, the right jump probability is $\frac{1}{2} - \epsilon$, and the left jump probability is $\frac{1}{2} + \epsilon$. Let T
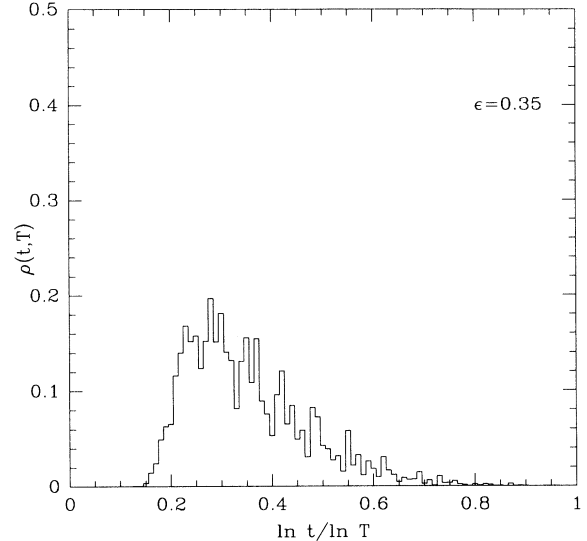

FIG. 1. Histogram of $\rho(t, T)$ in terms of the scaled variable $\ln t / \ln T$ for $\epsilon = 0.35$, $N = 20$.

denote the largest value of t,

$$T = \frac{(1 + \gamma)}{1 - \gamma} N + \frac{\gamma(1 + \gamma)}{(1 - \gamma)^2}(\gamma^N - 1), \qquad (3)$$

where $\gamma(\epsilon) = (1 + 2\epsilon)/(1 - 2\epsilon) > 1 \quad \forall \quad \epsilon > 0$. Asymptotically $(N \to \infty)$, we see that $T \sim \gamma^N(\epsilon)$.

Let $\rho(t, T)$ denote the distribution of MFPT over the disorder. In Fig. 1 we show a histogram plot of $\rho(t, T)$ where we have binned the 2^N values of t in equal intervals of the scaled variable $\ln t / \ln T$. The figure corresponds to $N = 20$ and $\epsilon = 0.35$. The distribution is broad. Previously this distribution was studied both numerically [6] and analytically by asymptotic methods [17]. Although there is qualitative agreement with the asymptotic prediction of Ref. [17], there is a quantitative difference indicating that the asymptotic limit is not yet reached.

We now turn our attention to the moments $M(Q, T)$ of $\rho(t, T)$ that were defined in the Introduction. In principle the moments could be calculated from the asymptotic distribution given in Ref. [17]; in practice this appears to be a difficult task. On the basis of results for known examples (see, e.g., Ref. [18]) we expect that the analysis of the moments for finite chains gives accurate predictions for the scaling analysis.

The first task would be to examine the power-law dependence of the moments on T, as implied in Eq. (1). This property holds for the model under investigation, as will be demonstrated later in the context of the partition function. Figure 2 then depicts the exponents $\xi(Q)$ that describe the the divergence of the moments, as a function of Q, for two cases, one with weak disorder $(\epsilon = 10^{-4})$, and the other with strong disorder $(\epsilon = 0.35)$. We have generalized to noninteger as well as negative values of Q. We observe that when ϵ is large, $\xi(Q)$ varies nonlinearly with Q; see Fig. 2(b). However, in the limit of weak disorder, we find that $\xi(Q)$ is linear in Q within the accuracy of the numerical simulations; see Fig. 2(a), and

a single gap exponent is adequate to describe the distribution. The large range of the values of ξ in Fig. 2(a) stems from the choice of a small value of ϵ, as can be seen from the expression for γ; see Eq. (9) below. Let $\rho(t,T)$ denote the limiting distribution when $T \to \infty$. The aim is to characterize the limiting distribution $\rho(t, T \to \infty)$, employing multifractal formalisms, and to this we turn our attention now.

III. MULTIFRACTAL ANALYSIS

Let us define a partition function as follows:

$$\hat{Z}(Q,T) = \sum_{i}^{2^N} t_i^Q(T) \equiv 2^N M(Q,T), \qquad (4)$$

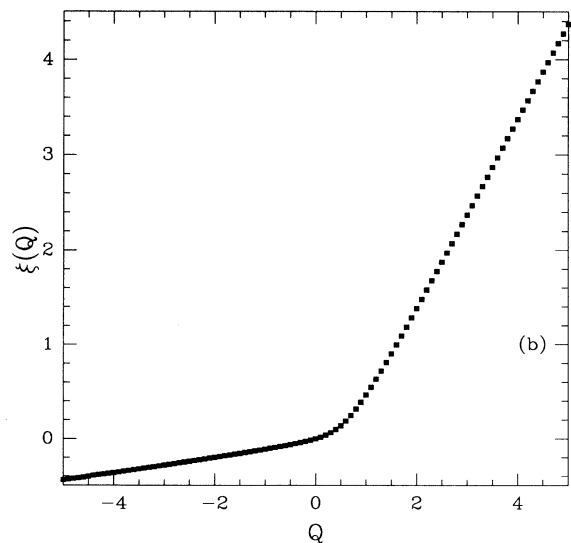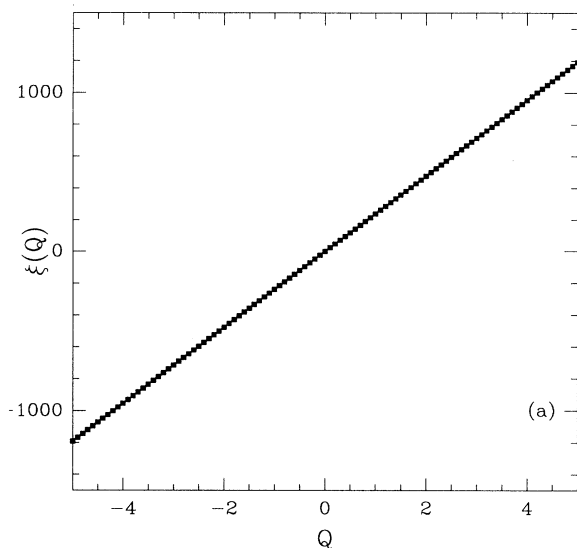where t_i are the MFPT in the 2^N different realizations

of the disorder, with $t_{min} \leq t_i \leq t_{max}$, and $-\infty < Q < +\infty$. Asymptotically, t_{min} diverges linearly with N. Note that the minimum value of t obtains when all the sites have right jump probabilities equal to $\frac{1}{2} + \epsilon$, and left jump probabilities equal to $\frac{1}{2} - \epsilon$. Also t_{max} is equal to T, which diverges exponentially with N.

To relate the moments to multifractal analysis, we consider the normalized partition function defined as

$$Z(Q,T) = \frac{\hat{Z}(Q,T)}{[\hat{Z}(Q=1,T)]^Q} \equiv 2^{-N(Q-1)} \frac{M(Q,T)}{[M(1,T)]^Q}. \qquad (5)$$

The normalized partition function can be written as

$$Z(Q,T) = \sum_{i}^{2^N} \hat{t}_i^Q, \qquad (6)$$

with the variable $\hat{t}_i = t_i / \sum_i t_i$. The reason for considering the normalized partition function now becomes obvious: The variables $\hat{t}_i$ correspond to the normalized measures that are used in conventional mutifractal analysis. Note that $Z(Q = 1,T) = 1$ because of normalization. The normalized partition function exhibits the customary behavior with T for $T \to \infty$, namely, it goes to zero for $Q > 1$ while it diverges for $Q < 1$.

We postulate that in the limit $T \to \infty$, the partition function obeys a scaling relation,

$$Z(Q,T) \sim T^{-\tau(Q)}. \qquad (7)$$

Figure 3 gives a log-log plot of the normalized partition function versus T for various values of Q, where we used $\epsilon = 0.25$ and N between 6 and 14. The points corresponding to each value of Q fall on a straight line, verifying the scaling ansatz. Also note that the different changes of the slope corresponding to the different values of Q indicate a nontrivial dependence of the scaling
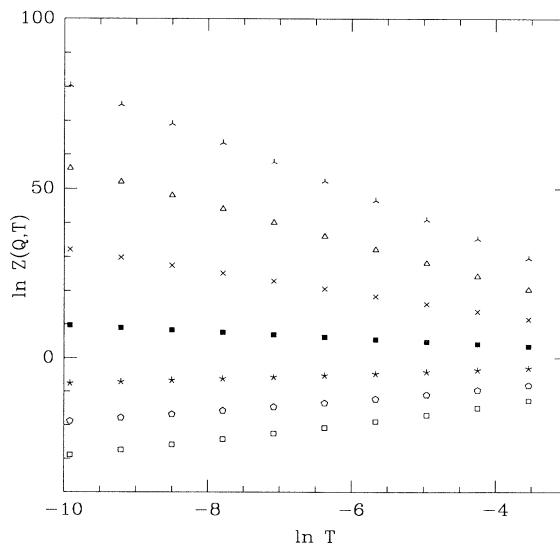

FIG. 2. $\xi(Q)$ vs Q for (a) $\epsilon = 10^{-4}$ and (b) $\epsilon = 0.35$, for $N = 6$ to 20.

FIG. 3. Plot of the normalized partition function $Z(Q,T)$ vs T on a log-log scale for values of Q ranging from -6 (top) to $+6$ (bottom) in units steps of 2. $\epsilon = 0.25$, $N = 6$ to 14.
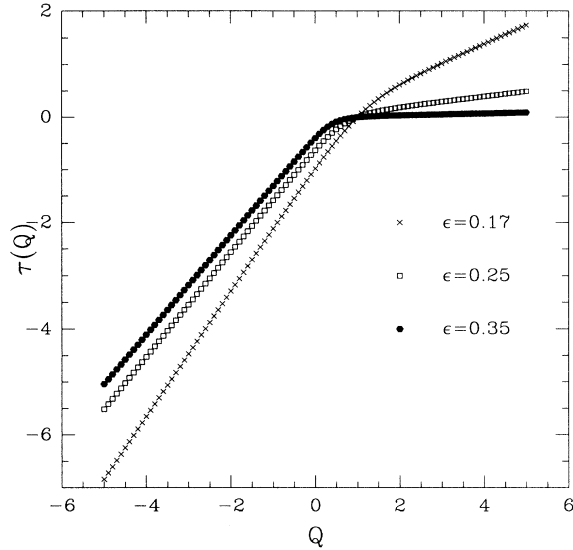
FIG. 4. $\tau(Q)$ vs Q for $\epsilon = 0.17$, 0.25, 0.35.

exponent on Q.

The multifractal structures associated with the limiting distribution of the MFPT are characterized [12] through the scaling exponents, the generalized Renyi dimensions, and the spectrum of singularities. The slopes of the log-log plot of the normalized partition function versus the system size, see Fig. 3, give the scaling exponents, $\tau(Q)$ and these are plotted in Fig. 4 for three representative values of ϵ. The nonlinear dependence of τ on Q, establishes unambiguously that the MFPT distribution is multifractal. Figure 5 depicts the generalized Renyi dimension spectra $D(Q) = \tau(Q)/(Q-1)$, for the same values of disorder. We find that $D(Q)$ is a monotonically nonincreasing function of Q. We take the Legendre transform of $\tau(Q)$, defined as $f(\alpha) = -\tau(Q) + q\alpha$, where α is the first derivative of τ with respect to Q.
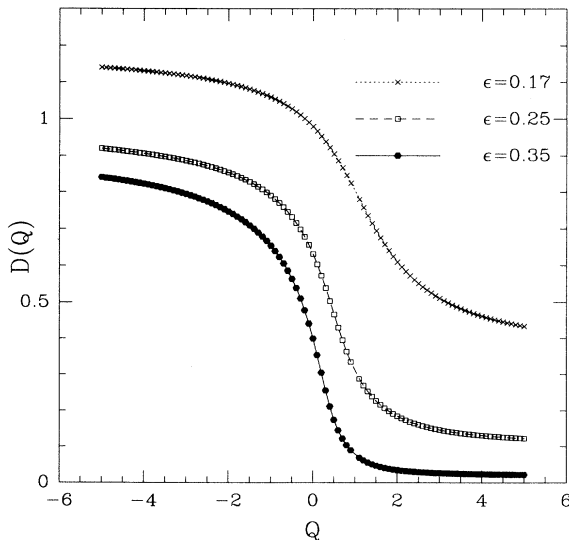

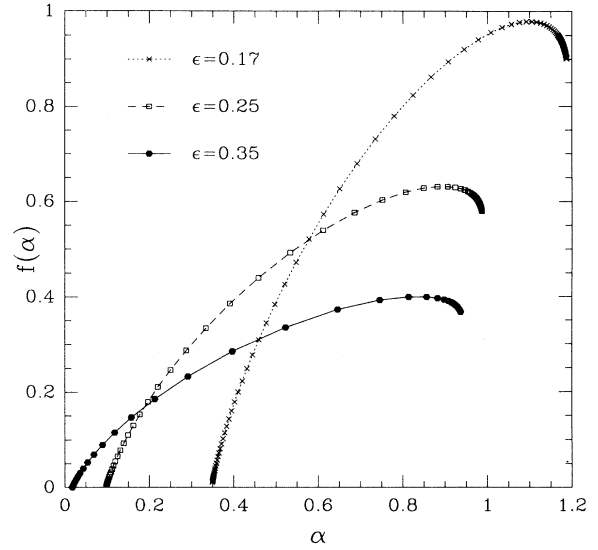FIG. 5. Generalized Renyi dimensions $D(Q)$ vs Q for $\epsilon = 0.17$, 0.25, 0.35.

FIG. 6. Spectrum of the singularities $f(\alpha)$ vs α for the same values of ϵ as in Fig. 5.

Figure 6 depicts the $f(\alpha)$ spectrum. The curve is convex with a single maximum at the value of $\alpha_0 \equiv \alpha(Q = 0)$, and $f(\alpha_0) = D(0)$. The different values of $f[\alpha(Q)]$ are the fractal dimensions of the subsets that contribute maximally to the normalized partition function as Q is varied from $-\infty$ to $+\infty$. The limiting values $f[\alpha(Q \to \pm\infty)]$, give the fractal dimensions of the densest and rarest regions of the distribution. We find that $f(\alpha_{min}) = 0 \quad \forall \quad \epsilon$, which implies that the densest region is a set of points with a fractal dimension 0. On the other hand $f(\alpha_{max})$ is nonzero and hence the rarest region is a fractal of dimension 0.37 for $\epsilon = 0.35$, and it increases to 0.9 when ϵ is decreased to 0.17.

The results of this section demonstrate that the standard multifractal analysis can indeed be made when working with the normalized moments. Further, they demonstrate that the distributions of MFPT in different realizations of the Sinai disorder constitute a multifractal measure.

IV. ANALYTICAL EXPRESSION FOR THE FRACTAL DIMENSION

We note that the exponent $\xi(1)$ characterizing the divergence of the first moment, $M(1, T)$, is exactly known [15] and is given by

$$\xi(1) = \frac{\ln\beta(\epsilon)}{\ln\gamma(\epsilon)}, \tag{8}$$

where $\beta(\epsilon)$ and $\gamma(\epsilon)$ have been defined earlier (see Sec. II). Comparing the definition of $\xi(Q)$ Eq. (1) with Eqs. (4,5) the relation between $\tau(Q)$ and $\xi(Q)$ can now be written as

$$\tau(Q) = -\xi(Q) + (Q-1)\frac{\ln 2}{\ln\gamma(\epsilon)} + Q\frac{\ln\beta(\epsilon)}{\ln\gamma(\epsilon)}. \tag{9}$$

The fact that $\tau(Q)$ goes like $-\xi(Q)$ is related to the

observation that $Z(Q,T) \sim \infty$, for $Q < 1$, whereas $M(Q,T) \sim \infty$ for $Q > 1$. Also notice the difference in the normalization: $Z(Q = 1, T) = M(Q = 0, T) = 1$. The explicit forms of the Q-dependent additive terms are specific to the phenomenon and the model under investigation, and hence would be different for different physical systems.

We immediately see that the expression for $\tau(Q = 0)$ is related to the divergence of the number of possible realizations of the Sinai lattice vis-à-vis the divergence of T, as $N \to \infty$. The former diverges as 2^N, while the latter diverges as $\gamma^N(\epsilon)$. Hence $D(0) = -\tau(0) = \ln(2)/\ln[\gamma(\epsilon)]$. When $\gamma(\epsilon)$ is larger than 2, T diverges faster than the number of values of t_i, and hence the distribution of the t_i's becomes sparser and sparser, as $N \to \infty$, leading to $D(0) < 1$. However, when $\gamma(\epsilon) = 2$, the number of possible values of t and T diverge at the same rate as $N \to \infty$, and we should expect the distribution to be space filling, implying that $D(0) = 1$. The value of ϵ_c for which $\gamma(\epsilon_c) = 2$ is 1/6. We have plotted $D(0) = \ln 2 / \ln \gamma(\epsilon)$ as a function of ϵ in Fig. 7. We have also shown in the same figure the results of $D(0)$ versus ϵ from an earlier numerical study [14], wherein the 2^N values of t were scaled and represented as dots on a unit line segment, and the fractal measures of the density distribution of these dots were obtained employing conventional methods. The results of our analytical prediction agree well with the earlier numerical calculations, especially for larger values of ϵ. The large deviation that we find for small values of ϵ, is due to finite-size effects in the earlier numerical work.

Thus, we observe, whenever we have distributions whose moments are all unbounded, and if the different moments scale differently with the scale parameter, we can expect underlying multifractal fluctuations. The exponents characterizing the fractal measures can in principle be derived from the exponents characterizing the scaling of the moments. The fractal dimension $D(0)$ is

obtained by observing that asymptotically $(N \to \infty)$ $T \sim \gamma^N$, whereas the number of disorder configurations diverges as 2^N.

V. SELF-SIMILAR FLUCTUATIONS OF MFPT

Intuitively one associates pictures of wildly varying probabilities with multifractal measures, as we move from one region of the support to regions infinitesimally close to it. This behavior was characterized as "interwoven sets of singularities" by Halsey et al. [13]. It appears as if this intuitive picture is lost when considering the distributions $\rho(t, T)$. To reconstitute the idea of a statistical multifractal, we consider the set of 2^N values of t_i that correspond to all the possible disorder configurations for a given ϵ and N. Note that each disorder configuration is equally probable and the probability is 2^{-N}. We generate a set of large, say M, number of integer random numbers independently from a uniform distribution between 1 and 2^N, and let this set be denoted by $\{\nu(i); i = 1, 2, \ldots M\}$. We plot $t_{\nu(i)}$ versus i, and Fig. 8 depicts one such plot that corresponds to $N = 14$, and $\epsilon = 0.35$. The sample size is about $M = 40\,000$. We see that the value of t fluctuates strongly from one realization to the other and these fluctuations constitute a statistical multifractal. To see the statistical self-similarity of these fluctuations, we have shown in the inset of Fig. 8 a portion of these fluctuations that corresponds to $i = 9000 - 17\,000$. The part and the whole are found to be statistically self-similar. The multifractal scaling exponents, the generalized dimensions, and the spectrum of singularities help to precisely capture the various nuances of this statistical self-similarity, inherent in the fluctuations of t from one random realization to the other.
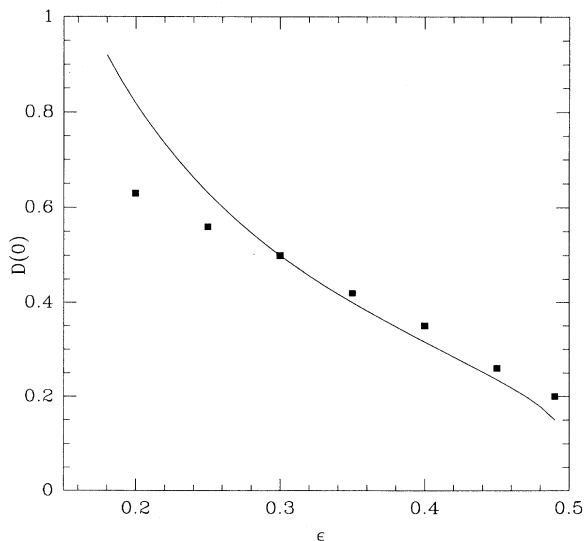

FIG. 7. Variation of $D(0)$ with ϵ. Points are the numerical results of Ref. [14], while the solid line is obtained from the analytical expression given in Sec. IV.
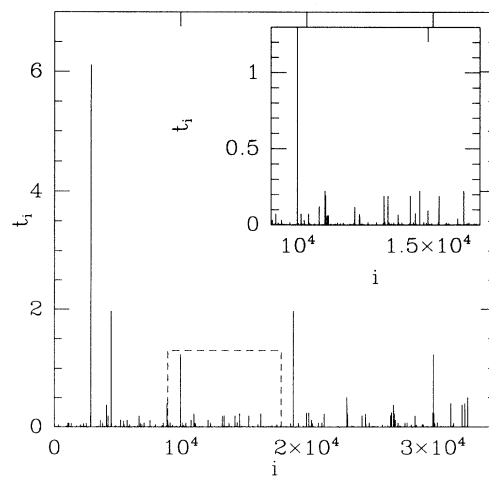

FIG. 8. Plot of the random variable $t_{\nu(i)}$ obtained from a random sampling procedure (see text) versus the index i. In the inset the portion contained within the box has been blown up to display the statistical self-similarity of the fluctuations of t.

VI. CONCLUSIONS

We have considered the distribution of MFPT over Sinai disorder by employing multifractal formalisms. We have characterized the multifractality of the distribution of the MFPT over Sinai disorder through the scaling exponents, the generalized Renyi dimensions, and the spectrum of singularities. The presence of multifractality in the distributions of the MFPT is obviously a consequence of the multiplicative structure of Eq. (2) for the individual MFPT. For the dichotomic model of the Sinai disorder, we find that the fractal dimension can be calculated analytically by considering the relative divergence of the number of possible disorder configurations and of the scale parameter as the number of lattice sites goes to infinity.

Although *in principle* the exponents characterizing the divergence of the physical moments and the multifractal exponents are related, derivation of the precise relation for a general problem is not an easy task. Here, however, we have been able to derive such a relation due the inherent simplifying features of the model considered. Hence we have been able to render transparent the nature and the content of fractal measures associated with the distribution of MFPT, which in the limit of large system sizes has all positive moments infinite.

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